APPROXIMATION AND DERIVATIVES OF SURVIVAL IN STRUCTURAL ANALYSIS AND DESIGN

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Abstract

Yield stresses, allowable stresses, moment capacities (plastic moments), external loadings, manufacturing errors,... are not given fixed quantities in practice, but have to be modelled as random variables with a certain joint probability distribution. Hence, problems from limit (collapse) load analysis or plastic analysis and from plastic and elastic design of structures are treated in the framework of stochastic optimization. Using especially reliability-oriented optimization methods, the behavioral constraints are quantified by means of the corresponding probability p_s of survival. Lower bounds for p_s are obtained by selecting certain redundants in the vector of internal forces/bending-moments; moreover, upper bounds for p_s are constructed by considering a pair of dual linear programs for the optimizational representation of the yield or safety constraints. Whereas the probability p_s can be computed e.g. by sampling methods or by asymptotic expansion techniques based on Laplace integral representations of certain multiple integrals, efficient techniques for the computation of the sensitivities (of various orders) of p_s with respect to input or design variables X and load factors λ have yet to be developed. Hence, several new techniques (e.g. Transformation Method, Stochastic Completion Technique) are suggested for the numerical computation of derivatives of p_s with respect to (X,λ).

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1. Limit (collapse) load analysis of structures as a linear programming problem.

The **collapse load** can be defined [3],[5],[18] "as the load required to generate enough number of local plastic yield points (referred as plastic hinges for bending type members) to cause the structure to become a mechanism and develop excessive deflections". Assuming that the material behaves as an elastic-perfectly plastic material [4],[15], a conservative estimate of the collapse load factor λ_T is based on the following formulation as a linear program (LP):

maximize λ (1)

s.t.

$$F^L \le F \le F^U \tag{1.1}$$

$$CF = \lambda R_o. \tag{1.2}$$

Here, (1.2) is the equilibrium equation of a statically indeterminate loaded structure involving an m×n matrix $C = (c_{ij})$, m<n, of given coefficients c_{ij}, $1 \le i \le m$, $1 \le j \le n$, depending on the undeformed geometry of the structure having n_o members (elements); we suppose that rankC=m. Furthermore, R_o is an external load m-vector, and F denotes the n-vector of internal forces and bending-moments in the relevant points (sections, nodes or elements) of the structure. Finally, (1.1) are the yield conditions with the vector of lower and upper bounds F^L, F^U.

For a **plane or spatial truss** [6],[16] we have that $n=n_o$, the matrix C contains the direction cosines of the members, and F involves only the normal (axial) forces, moreover,

$$F_j^L := \sigma_{yj}^L A_j,\ F_j^U := \sigma_{yj}^U A_j,\ j=1,\dots,n(=n_o), \tag{1.3}$$

where A_j is the (given) cross-sectional area, and σ_{yj}^L, σ_{yj}^U, resp., denotes the yield stress in compression (negative values) and tension (positive values) of the j-th member of the truss. In case of a **plane frame**, F is composed of subvectors [16]

$$F^{(k)} = \begin{pmatrix} F_1^{(k)} \\ F_2^{(k)} \\ F_3^{(k)} \end{pmatrix} = \begin{pmatrix} t_k \\ m_k^+ \\ m_k^- \end{pmatrix}, \tag{1.4}$$

where $F_1^{(k)}=t_k$ denotes the normal (axial) force, and $F_2^{(k)}=m_k^+$, $F_3^{(k)}=m_k^-$ are the bending-moments at the positive, negative end of the k-th member. In this case F^L, F^U contain - for each member k - the subvectors

$$F^{(k)L} = \begin{pmatrix} \sigma_{yk}^L A_k \\ -M_{kpl} \\ -M_{kpl} \end{pmatrix}, \quad F^{(k)U} = \begin{pmatrix} \sigma_{yk}^U A_k \\ M_{kpl} \\ M_{kpl} \end{pmatrix}, \tag{1.4.1}$$

resp., where M_{kpl}, $k=1,\dots,n_o$, denote the plastic moments (moment capacities) given [4],[15] by

$$M_{kpl} = \sigma_{yk}^U W_{kpl}, \tag{1.4.2}$$

and $W_{kpl} = W_{kpl}(A_k)$ is the plastic section modulus of the cross-section of the k-th member (beam) with respect to the local z-axis.

For a **spatial frame** [6],[16], corresponding to the k-th member (beam), F contains the subvector

$$F^{(k)} := (t_k, m_{kT}, m_{k\bar{y}}^+, m_{k\bar{z}}^+, m_{k\bar{y}}^-, m_{k\bar{z}}^-)', \tag{1.5}$$

where t_k is the normal (axial) force, m_{kT} the twisting moment, and $m_{k\bar{y}}^+$, $m_{k\bar{z}}^+, m_{k\bar{y}}^-, m_{k\bar{z}}^-$ denote four bending moments with respect to the local $\bar{y}$-,$\bar{z}$-axis at the positive, negative end of the beam, respectively. Finally, corresponding to (1.3),(1.4)-(1.4.2), the bounds F^L, F^U for F are given by

$$F^{(k)L} = (\sigma_{yk}^L A_k, -M_{kpl}^{\bar{p}}, -M_{kpl}^{\bar{y}}, -M_{kpl}^{\bar{z}}, -M_{kpl}^{\bar{y}}, -M_{kpl}^{\bar{z}})' \tag{1.5.1}$$

$$F^{(k)U} = (\sigma_{yk}^U A_k, M_{kpl}^{\bar{p}}, M_{kpl}^{\bar{y}}, M_{kpl}^{\bar{z}}, M_{kpl}^{\bar{y}}, M_{kpl}^{\bar{z}})', \tag{1.5.2}$$

where, cf. [4],[15],

$$M_{kpl}^{\bar{p}} := \tau_{yk} W_{kpl}^{\bar{p}}, \quad M_{kpl}^{\bar{y}} := \sigma_{yk}^U W_{kpl}^{\bar{y}}, \quad M_{kpl}^{\bar{z}} := \sigma_{yk}^U W_{kpl}^{\bar{z}} \tag{1.5.3}$$

are the plastic moments of the cross-section of the k-th element with respect to the local twisting axis, the local $\bar{y}$-,$\bar{z}$-axis, respectively. In (1.5.3), $W_{kpl}^{\bar{p}}=W_{kpl}^{\bar{p}}(X)$ and $W_{kpl}^{\bar{y}}=W_{kpl}^{\bar{y}}(X)$, $W_{kpl}^{\bar{z}}=W_{kpl}^{\bar{z}}(X)$, resp., denote the polar, axial modulus of the cross-sectional area of the k-th beam and τ_{yk} denotes the yield stress with respect to torsion; we suppose $\tau_{yk}=\sigma_{yk}^U$.

Remark 1.1

Possible **plastic hinges** [4],[7],[15] are taken into account by inserting appropriate eccentricities $e_{kl}>0, e_{kr}>0$, $k=1,\dots,n_o$, with $e_{kl}, e_{kr} \ll L_k$, where L_k is the length of the k-th beam.

Remark 1.2

Working with more general **yield polygons** [1],[17],[18], the stress condition (1.1) is replaced by the more general system of inequalities

$$H(F_d^U)^{-1}F \le h. \tag{1.1a}$$

Here, (H,h) is a given $\nu\times(n+1)$ matrix, and $F_d^U:=(F_j^U\delta_{ij})$ denotes the $n\times n$

diagonal matrix of **principal axial and bending plastic capacities**

$$F_j^U := \sigma_{yk_j}^U A_{k_j}, \quad F_j^U := \sigma_{yk_j}^U W_{k_j}^{\tau_j} p_1,$$

where k_j, τ_j are indices as arising in (1.4.1)-(1.5.3). The more general case (1.1a) can be treated by similar methods as the case (1.1) which is considered here.

2. Plastic and elastic design of structures

In the plastic design of trusses, frames [5] having n_o members, the n-vectors F^L, F^U of lower and upper bounds

$$F^L = F^L(\sigma_y^L, \sigma_y^U, X), \quad F^U = F^U(\sigma_y^L, \sigma_y^U, X) \tag{2}$$

for the n-vector F of internal member forces and bending-moments F_j, j=1,...,n, are determined [3],[5] by the yield stresses, i.e. compressive limiting stresses (negative values) $\sigma_y^L = (\sigma_{y1}^L, \dots, \sigma_{yn_o}^L)'$, the tensile yield stresses $\sigma_y^U = (\sigma_{y1}^U, \dots, \sigma_{yn_o}^U)'$, and the r-vector

$$X = (X_1, X_2, \dots, X_r)' \tag{2.1}$$

of design variables of the structure. In case of trusses we simply have that, cf. (1.3),

$$F^L = \sigma_{yd}^L A(X) = A(X)_d \sigma_y^L, \quad F^U = \sigma_{yd}^U A(X) = A(X)_d \sigma_y^U, \tag{2.2}$$

where $n=n_o$, and $\sigma_{yd}^L, \sigma_{yd}^U$ denote the n×n diagonal matrices having the diagonal elements $\sigma_{yj}^L, \sigma_{yj}^U$, resp., j=1,...,n; moreover,

$$A(X) = (A_1(X), \dots, A_n(X))' \tag{2.3}$$

is the n-vector of cross-sectional areas $A_j = A_j(X)$, j=1,...,n, depending on the r-vector X of design variables X_k, k=1,...,r, and $A(X)_d$ denotes the n×n diagonal matrix having the diagonal elements $A_j = A_j(X)$, $1 \le j \le n$.

Corresponding to (1.2), here we have again the equilibrium equation

$$CF = R_u, \tag{2.4}$$

where R_u describes [5] the ultimate load (representing constant external loads or self-weight expressed in linear terms of A(X)).

The **plastic design** of structures can be represented then [1],[2],[5] by the following optimization problem

$$\min G(X) \tag{3}$$

s.t.

$$F^L(\sigma_y^L, \sigma_y^U, X) \le F \le F^U(\sigma_y^L, \sigma_y^U, X) \tag{3.1}$$

$$CF = R_u, \tag{3.2}$$

where G=G(X) is a certain objective function, e.g. the volume or weight of the structure.

Remark 2.1

As mentioned in **Remark 1.2**, working with more general yield polygons, (3.1) is replaced by the condition

$$H(F^U(\sigma_y^U,X)_d)^{-1}F \le h. \tag{3.1a}$$

For the **elastic design** we have to replace the yield stresses σ_y^L,σ_y^U by the allowable stresses σ_a^L,σ_a^U, and instead of ultimate loads we consider service loads R_s. Hence, instead of (3) we get the related program

$$\min G(X) \tag{4}$$

s.t.

$$F^L(\sigma_a^L,\sigma_a^U,X) \le F \le F^U(\sigma_a^L,\sigma_a^U,X) \tag{4.1}$$

$$CF = R_s \tag{4.2}$$

$$X^L \le X \le X^U, \tag{4.3}$$

where X^L,X^U still denote lower and upper bounds for X.

3. Analysis and design of structures in case of random data

In practice, yield stresses, allowable stresses, the loads applied to the structure, other material properties and the manufacturing errors are not given fixed quantities, but must be treated as random variables on a certain probability space $(\Omega,\mathcal{A},P)$. Hence, (1),(3),(4) are stochastic linear/ nonlinear programs (SLP/SNLP) which obviously have the same basic structure represented by a random objective function

$$Z(\omega) := G(\omega,X),\ \omega \in \Omega, \tag{5}$$

and by stochastic constraints of the type

$$CF = R(\omega) \tag{6}$$

$$F^L(\omega) \le F \le F^U(\omega), \tag{7}$$

where $R=R(\omega)$, $\omega \in \Omega$, is a random load m-vector given by

$$R(\omega) = \lambda R_o(\omega),\ R(\omega) = R_u(\omega),\ R(\omega) = R_s(\omega), \tag{6.1}$$

resp., and for the n-vector $F=(F_j)$ of internal member forces and bending-moments we have the n-vectors of random bounds

$$F^L(\omega) = F^L(\omega,X),\ F^U(\omega) = F^U(\omega,X),\ \omega \in \Omega, \tag{7.1}$$

depending on an r-vector X of design variables X_k, k=1,...,r.

Obviously, each realization of the random element $\omega\in\Omega$ yields new loading conditions, represented by the vector $R=R(\omega)$, and therefore, cf. (6), new arrangements $F=F(\omega)$ of internal member forces and bending-moments. Hence, in the present case of "multiple loadings" caused by the random variations of $R=R(\omega)$, the survival of the structure, i.e. the existence of certain arrangements $F(\omega)$ of internal member forces/bending-moments not overwhelming the strength of the structure, can be evaluated by the probability of survival

$$p_s := P(\text{There is } F=F(\omega) \text{ such that } CF(\omega) = R(\omega) \text{ and } F^L(\omega) \le F(\omega) \le F^U(\omega)), \tag{8}$$

see [2],[14], assuming that, cf. (26),(33),

$$S(F^L(.),F^U(.),R(.)) := \{\omega\in\Omega: \text{ there is a vector } F=F(\omega) \text{ fulfilling (6) and (7)}\} \tag{8.1}$$

is a measurable set. Denoting by $[F,^L,F^U]$ the n-dimensional interval

$$[F^L,F^U] := \{F\in\mathbb{R}^n: F^L \le F \le F^U\}, \tag{8.2}$$

we find that

$$S(F^L(.),F^U(.),R(.)) = \{\omega\in\Omega: R(\omega) \in C[F^L(\omega),F^U(\omega)]\} \tag{8.3}$$

with $C[F^L,F^U] = \{CF: F^L \le F \le F^U\}$, and therefore

$$\begin{aligned} p_s &= P(S(F^L(.),F^U(.),R(.))) = P(R(\omega)\in C[F^L(\omega),F^U(\omega)]) \\ &= \int P(R(\omega)\in C[F^L,F^U] \mid F^L(\omega)=F^L, F^U(\omega)=F^U)\pi(dF^L,dF^U), \end{aligned} \tag{9}$$

where π denotes the distribution of the bounds $(F^L(\omega),F^U(\omega))$. Since the bounds F^L,F^U in (7) depend also on the vector X of design variables X_k, $k=1,\dots,r$, cf. (7.1), we have $p_s=P(X)$ with the probability function

$$P(X) = P(R(\omega) \in C[F^L(\omega,X),F^U(\omega,X)]); \tag{10}$$

furthermore, if the external load $R=R(\omega)$ is given by

$$R(\omega) = R(\omega,\lambda) := \sum_{i=1}^{m_R} \lambda_i R^{(i)}(\omega), \quad \lambda := (\lambda_1,\dots,\lambda_{m_R})', \tag{10.1}$$

with random m-vectors $R^{(i)} = R^{(i)}(\omega)$, $i=1,\dots,m_R$, and deterministic coefficients λ_i, $i=1,\dots,m_R$, then $p_s=P(X,\lambda)$ with

$$P(\lambda,X) := P(\sum_{i=1}^{m_R} \lambda_i R^{(i)}(\omega) \in C[F^L(\omega,X),F^U(\omega,X)]). \tag{10.2}$$

Especially, in case of trusses, see (1),(3),(4) and (2.2), for dealing with the probability of survival p_s we have the following probability functions:

$$P_A(\lambda,X) := P(\lambda R_o(\omega) \in C[A(X)_d\sigma_y^L(\omega),A(X)_d\sigma_y^U(\omega)]), \quad \lambda\in\mathbb{R} \tag{10.3}$$

$$P_u(X) := P(R_u(\omega) \in C[A(X)_d\sigma_y^L(\omega),A(X)_d\sigma_y^U(\omega)]) \tag{10.4}$$

$$P_s(X) := P(R_s(\omega) \in C[A(X)_d\sigma_a^L(\omega),A(X)_d\sigma_a^U(\omega)]). \tag{10.5}$$

Since $p_s=P(X)$ are very complicated expressions in general, in the following we are looking for approximations (lower, upper bounds) of $p_s=P(X)$ by simpler probability functions.

4. Lower and upper bounds for $p_s = P(X)$

According to (8.3) we have that

$$S(F^L(.),F^U(.),R(.)) \subset \bigcap_{i=1}^{m} S_i(F^L(.),F^U(.)), \tag{11}$$

where

$$S_i(F^L(.),F^U(.),R(.)) := \{\omega\in\Omega:\ R_i(\omega) \in C_i[F^L(\omega),F^U(\omega)]\},\ i=1,\dots,m, \quad (11.1)$$

and C_i denotes the i-th row of C. Hence, we get the inequality

$$P(X) \le \min_{1\le i\le m} P_i(X), \tag{12}$$

where

$$P_i(X) := P(R_i(\omega) \in C_i[F^L(\omega,X),F^U(\omega,X)]). \tag{12.1}$$

Note that related inequalities (with more exact bounds) follow from more general Bonferroni-type inequalities [10]. We find that

$$C_i[F^L(\omega,X),F^U(\omega,X)] = [\gamma_i^L(\omega,X),\gamma_i^U(\omega,X)] \tag{12.2}$$

is an interval in $\mathbb{R}^1$ having the bounds

$$\begin{aligned}\gamma_i^L(\omega,X) &= \min_{1\le\iota\le J} C_i G^\iota(\omega,X)\\ \gamma_i^U(\omega,X) &= \max_{1\le\iota\le J} C_i G^\iota(\omega,X),\end{aligned} \tag{12.3}$$

where $G^\iota=G^\iota(\omega,X)$, $\iota=1,\dots,J$, are the extreme points of the interval $[F^L(\omega,X),F^U(\omega,X)]$. Since the components $G_j^\iota(\omega,X)$, $j=1,2,\dots,n$, of $G^\iota(\omega,X)$, $\iota=1,\dots,J$, are certain elements of $(F^L(\omega,X),F^U(\omega,X))$, the measurability of the bounds F^L,F^U with respect to $\omega\in(\Omega,\mathfrak{A},P)$ yields the measurability of γ_i^L and γ_i^U, $i=1,\dots,m$, with respect to ω. Hence, $S_i(F^L(.),F^U(.),R(.))$, $i=1,\dots,m$, are measurable sets, and we may write

$$P_i(X) = P(\gamma_i^L(\omega,X) \le R_i(\omega) \le \gamma_i^U(\omega,X)),\ i=1,\dots,m. \tag{12.4}$$

4.1. Lower bounds by selection of redundants

For the construction of lower bounds for P(X), the vector $F=F(\omega)$ is partitioned

$$F(\omega) = \begin{pmatrix} F_I \\ N \end{pmatrix} \tag{13}$$

into a certain (n-m)-vector $N=(F_{j_\ell})_{1\le\ell\le n-m}$ of **redundants** F_{j_ℓ}, $\ell=1,\dots,n-m$, and an m-vector F_I of statically determined member forces/bending-moments. Hence, with a corresponding partition of the m×n matrix C into m×m, m×(n-m) submatrices C_I, C_{II}, resp., where

$$\mathrm{rank} C_I = \mathrm{rank} C = m, \tag{13.1}$$

the equilibrium equation (6) yields for $F(\omega)$ the representation

$$F(\omega) = \begin{pmatrix} F_I \\ N \end{pmatrix} = \begin{pmatrix} C_I^{-1}R(\omega) \\ 0 \end{pmatrix} + \begin{pmatrix} -C_I^{-1}C_{II} \\ I \end{pmatrix} N. \tag{14}$$

Consequently, selecting for each $\omega\in\Omega$ a vector of redundants $N=N(\omega)=(F_{j_\ell}(\omega))_{1\le\ell\le n-m}$ such that N(.) is a measurable function on $(\Omega,\mathfrak{A},P)$, we get

$$S(F^L(.,X),F^U(.,X),R(.)) \supset \tilde{S}(X,N(.),R(.)), \tag{15}$$

where $\tilde{S}(X,N(.),R(.))$ is the measurable set given by

$$\tilde{S}(X,N(.),R(.)) := \{\omega\in\Omega:\ F^L(\omega,X) \le \begin{pmatrix} C_I^{-1}(R(\omega)-C_{II}N(\omega)) \\ N(\omega) \end{pmatrix} \le F^U(\omega,X)\}. \tag{15.1}$$

Thus, for P(X) we find the lower bound

$$P(X) \geq \tilde{P}(X,N(.)), \tag{16}$$

where

$$\tilde{P}(X,N(.)) := P\begin{pmatrix} F_I^L(\omega,X) \leq C_I^{-1}(R(\omega) - C_{II}N(\omega)) \leq F_I^U(\omega,X) \\ F_{II}^L(\omega,X) \leq N(\omega) \leq F_{II}^U(\omega,X) \end{pmatrix}, \tag{16.1}$$

and F_I^L, F_{II}^L and F_I^U, F_{II}^U denotes the partition of F^L, F^U, resp., corresponding to (13). Note that the inequality (16) holds for any choice $N=(F_{j_\ell})_{1\leq\ell\leq n-m}$ of an (n-m)-subvector of redundants such that (13.1) holds and any representation of N as a random vector $N=N(\omega)$ on $(\Omega,\mathfrak{A},P)$; especially, N can be selected as a **deterministic** vector of redundants:

$$N(\omega) = z \text{ a.s. (almost sure)}, \tag{17}$$

where $z\in\mathbb{R}^{n-m}$ is a deterministic vector; in this case we set $\tilde{P}(X,N(.)) = \tilde{P}(X,z)$.

4.1.1. Special cases. a) In case of trusses, cf. (1.3),(2.2), we have that

$$\begin{aligned} F_I^L(\omega,X) &= A_I(X)_d\sigma_I^L(\omega), \quad F_I^U(\omega,X) = A_I(X)_d\sigma_I^U(\omega) \\ F_{II}^L(\omega,X) &= A_{II}(X)_d\sigma_{II}^L(\omega), \quad F_{II}^U(\omega,X) = A_{II}(\omega)_d\sigma_{II}^U(\omega), \end{aligned} \tag{18}$$

where $A_I, A_{II}, \sigma_I^L, \sigma_{II}^L, \sigma_I^U, \sigma_{II}^U$ are the partitions of A, σ^L, σ^U, resp., corresponding to the partition F_I, F_{II} of F, and $A_I(X)_d$ denotes the diagonal matrix which has the components of $A_I(X)$ as its diagonal elements. Thus, we get

$$\begin{aligned} \tilde{P}(X,z) &= P\begin{pmatrix} A_I(X)_d\sigma_I^L(\omega) \leq C_I^{-1}(R(\omega) - C_{II}z) \leq A_I(X)_d\sigma_I^U(\omega) \\ A_{II}(X)_d\sigma_{II}^L(\omega) \leq z \leq A_{II}(X)_d\sigma_I^U(\omega) \end{pmatrix} \\ &= P\begin{pmatrix} \sigma_I^L(\omega) \leq A_I(X)_d^{-1}C_I^{-1}(R(\omega) - C_{II}z) \leq \sigma_I^U(\omega) \\ A_{II}(X)_d\sigma_{II}^L(\omega) \leq z \leq A_{II}(X)_d\sigma_{II}^U(\omega) \end{pmatrix}. \end{aligned} \tag{18.1}$$

b) Suppose that the partition of F^L, F^U into F_I^L, F_I^U and F_{II}^L, F_{II}^U can be chosen such that $(F_I^L(\omega,X), F_I^U(\omega,X))$, $(F_{II}^L(\omega), F_{II}^U(\omega))$ are stochastically independent. If (17) holds, then

$$\begin{aligned} \tilde{P}(X,z) := {} & P(F_I^L(\omega,X) \leq C_I^{-1}(R(\omega) - C_{II}z) \leq F_I^U(\omega,X)) \\ & \times P(F_{II}^L(\omega,X) \leq z \leq F_{II}^U(\omega,X)). \end{aligned} \tag{18.2}$$

5. Failure modes, limit state functions and upper bounds for P(X)

According to (8),(10) we have that

$$P(X) = P(\text{There is } F=F(\omega) \text{ such that } CF(\omega)=R(\omega) \text{ and } \begin{array}{l} F_j(\omega) - F_j^U(\omega,X) \le 0,\ j=1,\dots,n \\ F_j^L(\omega,X) - F_j(\omega) \le 0,\ j=1,\dots,n), \end{array} \tag{19}$$

where we suppose that all bounds F_j^L, F_j^U are finite, i.e.

$$-\infty < F_j^L(\omega,X) \le F_j^U(\omega,X) < +\infty \text{ a.s., } 1\le j\le n, \text{ for all } X \text{ under consideration.} \tag{19.1}$$

Defining

$$t(\omega,F(\omega),X) := \max_{1\le j\le n} \{F_j(\omega)-F_j^U(\omega,X), F_j^L(\omega,X)-F_j(\omega)\}, \tag{19.2}$$

we obtain

$$\begin{aligned} P(X) &= P(\text{There is } F=F(\omega) \text{ such that } CF(\omega)=R(\omega) \text{ and } t(\omega,F(\omega),X)\le 0) \\ &= P(\inf\{t(\omega,F(\omega),X):\ CF(\omega)=R(\omega)\} \le 0) \\ &= P(t^*(\omega,X) \le 0), \end{aligned} \tag{20}$$

where

$$t^*(\omega,X) := \inf\{t(\omega,F(\omega),X):\ CF(\omega)=R(\omega)\} \tag{20.1}$$

is the minimal value of the program

$$\min\ t(\omega,F(\omega),X) \tag{20.2}$$

s.t.

$$CF(\omega) = R(\omega)$$

being equivalent to the **linear program**

$$\text{minimize } t \tag{21}$$

s.t.

$$F_j - F_j^U(\omega,X) - t \le 0,\ j=1,\dots,n \tag{21.1}$$

$$F_j^L(\omega,X) - F_j - t \le 0,\ j=1,\dots,n \tag{21.2}$$

$$CF(\omega) = R(\omega) \tag{21.3}$$

with the variables $F_1, F_2, \dots, F_n, t$.

Because of condition (19.1), for each (ω,X) we get

$$t(\omega,F(\omega),X) \ge \max_{1\le j\le n} \frac{1}{2}(F_j^L(\omega,X) - F_j^U(\omega,X)) > -\infty \tag{21.4}$$

for arbitrary $F(\omega)$; hence, the objective function of the linear program (21) is bounded from below for all (ω,X). Since the LP (21) always has a feasible solution, for each (ω,X) an optimal solution $\binom{F^*}{t^*}$ of (21) is guaranteed, and we have that

$$t^* = t(\omega,F^*(\omega),X) = t^*(\omega,X). \tag{22}$$

Consequently, by means of duality theory the optimal value $t^*(\omega,X)$ of the equivalent programs (20.2) and (21) can be represented also by the optimal value of the dual program of (21) given by

$$\max R(\omega)'u - F^U(\omega,X)'\tilde{u}^+ + F^L(\omega,X)'\tilde{u}^- \tag{23}$$

s.t.

$$C'u - \tilde{u}^+ + \tilde{v}^- = 0 \tag{23.1}$$

$$1'\tilde{u}^+ + 1'\tilde{u}^- = 1 \tag{23.2}$$

$$\tilde{u}^+ \geq 0,\ \tilde{v}^- \geq 0, \tag{23.3}$$

where $u \in \mathbb{R}^n$ is not restricted.

Remark 5.1

Obviously, (23.1) is the member-node(or joint) displacement equation. According to its mechanical meaning, we call (21),(23), resp., the **static**, **kinematic** linear program (LP), cf. [2],[14].

Having (23), $t^*(\omega,X)$ reads

$$t^*(\omega,X) = \max\{\begin{pmatrix} R(\omega) \\ -F^U(\omega,X) \\ F^L(\omega,X) \end{pmatrix}' \delta : \begin{pmatrix} u \\ \tilde{u}^+ \\ \tilde{v}^- \end{pmatrix} =: \delta \in \Delta_o\}, \tag{24}$$

where Δ_o denotes the convex polyhedron in $\mathbb{R}^{m+2n}$ represented by the constraints (23.1)-(23.3) of the LP (23). Taking any subset $\Delta_1 \subset \Delta_o$ of Δ_o, and defining then $t_1^*(\omega,X)$ by

$$t_1^*(\omega,X) := \sup\{\begin{pmatrix} R(\omega) \\ -F^U(\omega,X) \\ F^L(\omega,X) \end{pmatrix}' \delta :\ \delta \in \Delta_1\}, \tag{25}$$

we get

$$t^*(\omega,X) \geq t_1^*(\omega,X), \tag{25.1}$$

which yields for P(X) the following upper bound:

$$P(X) = P(t^*(\omega,X) \leq 0) \leq P(t_1^*(\omega,X) \leq 0). \tag{25.2}$$

Moreover, if

$$\delta^{(\ell)} = \begin{pmatrix} u^{(\ell)} \\ \tilde{u}^{+(\ell)} \\ \tilde{u}^{-(\ell)} \end{pmatrix},\ \ell=1,\dots,\ell_o,$$

denote the extreme points of the convex polyhedron Δ_o, then

$$t^*(\omega,X) = \max_{1\leq\ell\leq\ell_o} R(\omega)'u^{(\ell)} - F^U(\omega,X)'\tilde{u}^{+(\ell)} + F^L(\omega,X)'\tilde{u}^{-(\ell)}, \tag{26}$$

which shows that $t^*(.,X)$ is measurable. Hence, $S(F^L(.),F^U(.),R(.)) = \{\omega\in\Omega: t^*(\omega,X)\leq 0\}$ is measurable, cf. (10),(20), and we get

$$P(X) = P(R(\omega)'u^{(\ell)} - F^U(\omega,X)'\tilde{u}^{+(\ell)} + F^L(\omega,X)'\tilde{u}^{-(\ell)} \leq 0,\ 1\leq\ell\leq\ell_o). \tag{26.1}$$

According to (6),(7), the survival, failure, resp., of the underlying structure can be described by the inequality

$$t^*(\omega,X) \leq 0,\ t^*(\omega,X) > 0, \text{ respectively.} \tag{27}$$

Thus, the structure fails if and only if

$$R(\omega)'u^{(\ell)} + F^U(\omega,X)'\tilde{u}^{+(\ell)} - F^L(\omega,X)'\tilde{u}^{-(\ell)} > 0 \text{ for at least one } 1\le\ell\le\ell_o; \tag{27.1}$$

obviously, (27.1) represents the different **failure modes** of the structure.

Having a certain number $\ell_1\le\ell_o$ of basic solutions $\delta^{(\ell_\tau)}$, $\tau=1,\dots,\ell_1$, of the LP (23), and defining

$$\tilde{t}_1^*(\omega,X) := \max_{1\le\tau\le\ell_1} R(\omega)'u^{(\ell_\tau)} - F^U(\omega,X)'\tilde{u}^{+(\ell_\tau)} + F^L(\omega,X)'\tilde{u}^{-(\ell_\tau)}, \tag{28}$$

corresponding to (25.1), here we get

$$t^*(\omega,X) \ge \tilde{t}_1^*(\omega,X) \tag{28.1}$$

and therefore

$$P(X) = P(t^*(\omega,X) \le 0) \le P(\tilde{t}_1^*(\omega,X) \le 0). \tag{28.2}$$

6. The probability of failure p_f

According to (8),(10),(20) and (27), for the probability of failure $p_f := 1-p_s = 1-P(X)$ we obtain

$$\begin{aligned} p_f &= P(t^*(\omega,X) > 0) \\ &= P\left(R(\omega)'u - F^U(\omega,X)'\tilde{u}^+ + F^L(\omega,X)'\tilde{u}^- > 0 \text{ for at least one} \begin{pmatrix} w \\ \tilde{u}^+ \\ \tilde{u}^- \end{pmatrix} \in \Delta_o\right) \\ &= P(R(\omega)'u^{(\ell)} - F^U(\omega,X)'\tilde{u}^{+(\ell)} + F^L(\omega,X)'\tilde{u}^{-(\ell)} > 0 \text{ for at least one } 1\le\ell\le\ell_o) \\ &= P\Big(\bigcup_{\ell=1}^{\ell_o} \mathbf{F}_\ell(X)\Big), \end{aligned} \tag{29}$$

where $\mathbf{F}_\ell(X)$ denotes the ℓ-th failure domain

$$\begin{aligned} \mathbf{F}_\ell(X) &:= \{\omega\in\Omega:\ R(\omega)'u^{(\ell)} - F^U(\omega,X)'\tilde{u}^{+(\ell)} + F^L(\omega,X)'\tilde{u}^{-(\ell)} > 0\} \\ &= \{\omega\in\Omega:\ M_\ell(\omega,X) < 0\} \end{aligned} \tag{29.1}$$

with the corresponding **limit state function**

$$M_\ell(\omega,X) := F^U(\omega,X)'\tilde{u}^{+(\ell)} - F^L(\omega,X)'\tilde{u}^{-(\ell)} - R(\omega)'u^{(\ell)}, \tag{29.2}$$

$\ell=1,\dots,\ell_o$; especially, for trusses, cf. (2.2), (18.1), we find

$$M_\ell(\omega,X) := \sigma^U(\omega)'A(X)_d\tilde{u}^{+(\ell)} - \sigma^L(\omega)'A(X)_d\tilde{u}^{-(\ell)} - R(\omega)'u^{(\ell)}. \tag{29.3}$$

Using known inequalities for probabilities, for p_f we find the bounds

$$\max_{1\le\ell\le\ell_o} p_{f,\ell} \le p_f \le \sum_{\ell=1}^{\ell_o} p_{f,\ell}, \tag{30}$$

where $p_{f,\ell}$ is given by

$$\begin{aligned} p_{f,\ell} &:= P(\mathbf{F}_\ell(X)) = P(M_\ell(\omega,X) < 0) \\ &= P(F^U(\omega,X)'\tilde{u}^{+(\ell)} - F^L(\omega,X)'\tilde{u}^{-(\ell)} < R(\omega)'u^{(\ell)}) \\ &= 1 - P(R(\omega)'u^{(\ell)} \le F^U(\omega,X)'\tilde{u}^{+(\ell)} - F^L(\omega,X)'\tilde{u}^{-(\ell)}), \end{aligned} \tag{30.1}$$

and sharper bounds can be obtained by using more genral Bonferroni-type inequalities for probabilities.

7. Representation of p_s by using cones

According to (9) we have that

$$p_s = P(R(\omega) \in C[F^L(\omega), F^U(\omega)]),$$

where $[F^L, F^U]$ is given by (8.2). Representing therefore the vector F of internal member forces/bending-moments by

$$F = F^L + \Delta F^L = F^U - \Delta F^U$$

with n-vectors $\Delta F^U, \Delta F^L \geq 0$, the condition $R \in C[F^L, F^U]$ can be represented by

$$\begin{aligned} R - CF^U &= -C\Delta F^U \\ -(F^U - F^L) &= -\Delta F^U - \Delta F^L \\ \Delta F^U \geq 0,\ \Delta F^L &\geq 0. \end{aligned} \tag{31}$$

Thus, we consider the cone $Y_o \subset \mathfrak{R}^{m+n}$ defined by

$$\begin{aligned} Y_o := &\left\{\begin{pmatrix} C & 0 \\ I & I \end{pmatrix}\begin{pmatrix} \Delta F^U \\ \Delta F^L \end{pmatrix}: \Delta F^U \geq 0,\ \Delta F^L \geq 0\right\} \\ = &\left\{\sum_{k=1}^{2n} \alpha_k y_k : \alpha_k \geq 0,\ k=1,\dots,2n\right\}, \end{aligned} \tag{32}$$

where the cone-generators $y^{(k)}$, $k=1,\dots,2n$, are given by

$$y^{(k)} := \begin{pmatrix} c_k \\ e_k \end{pmatrix},\ 1 \leq k \leq n,\quad y^{(k)} := \begin{pmatrix} 0 \\ e_k \end{pmatrix},\ n < k \leq 2n, \tag{32.1}$$

and c_k, e_k denotes the k-th column of C, of the n×n identity matrix I, respectively. Having Y_o, the set $S(F^L(.), F^U(.), R(.))$ defined in (8.1) can be described by

$$S(F^L(.), F^U(.), R(.)) = \left\{\omega \in \Omega: \begin{pmatrix} R(\omega) - CF^U(\omega, X) \\ -F^U(\omega, X) + F^L(\omega, X) \end{pmatrix} \in (-1)Y_o\right\}, \tag{33}$$

which shows again the measurability of $S(F^L(.), F^U(.), R(.))$.
Moreover, the probability function $P = P(\lambda, X)$ representing (10)-(10.3) can be given by

$$P(\lambda, X) = P\left(\begin{pmatrix} CF^U(\omega, X) - \lambda R_o(\omega) \\ F^U(\omega, X) - F^L(\omega, X) \end{pmatrix} \in Y_o\right). \tag{33.1}$$

Remark 7.1

The cone Y_o contains all pairs $(C\Delta F^U, \Delta F^L + \Delta F^U)$ of **admissible** pairs of external load/strength increments.

According to the representation (32) of Y_o, there are a finite number of boundary hyperplanes in $\mathfrak{R}^{m+n}$, represented by vectors $\eta^{(\ell)} = \begin{matrix} w^{(\ell)} \\ v^{(\ell)} \end{matrix}$, $\ell = 1, \dots, \ell_o'$, such that

$$Y_o = \left\{y = \binom{w}{v} \in \mathfrak{R}^{m+n}: y'\eta^{(\ell)} = w'w^{(\ell)} + v'v^{(\ell)} \geq 0,\ 1 \leq \ell \leq \ell_o'\right\}. \tag{34}$$

Hence, because of (33) and (34), the survival of the structure can be

represented also by the inequalities

$$(R(\omega)-CF^U(\omega,X))'w^{(\ell)}+(-F^U(\omega,X)+F^L(\omega,X))'v^{(\ell)}\le 0,\ 1\le\ell\le\ell_o',$$

which yields

$$(R(\omega)'w^{(\ell)}-F^U(\omega,X)'(C'w^{(\ell)}+v^{(\ell)})+F^L(\omega,X)'v^{(\ell)}\le 0,\ 1\le\ell\le\ell_o', \tag{35}$$

and therefore

$$p_s=P(R(\omega)'w^{(\ell)}-F^U(\omega,X)'(C'w^{(\ell)}+v^{(\ell)})+F^L(\omega,X)'v^{(\ell)}\le 0,\ 1\le\ell\le\ell_o'). \tag{35.1}$$

Obviously, the conditions for structural safety given by (27) and (35) coincide.

For arbitrary subset $Y_o^{(\ell)}$, $\ell=1,2$, such that

$$Y_o^{(1)}\subset Y_o\subset Y_o^{(2)}, \tag{36}$$

we get

$$p_s^{(1)}\le p_s\le p_s^{(2)}, \tag{36.1}$$

where the bounds $p_s^{(\ell)}$, $\ell=1,2$, are defined by

$$p_s^{(\ell)}:=P\left(\begin{pmatrix} CF^U(\omega,X)-R(\omega)\\ F^U(\omega,X)-F^L(\omega,X)\end{pmatrix}\in Y_o^{(\ell)}\right),\ \ell=1,2. \tag{36.2}$$

7.1. Construction of approximating cones

Suppose next to that we have a cone axis (center or middle line)

$$g=\{\lambda\bar{y}:\ \lambda\ge 0\} \tag{37}$$

generated by a vector $\bar{y}\in Y_o$, $\bar{y}\neq 0$, which is defined later on.

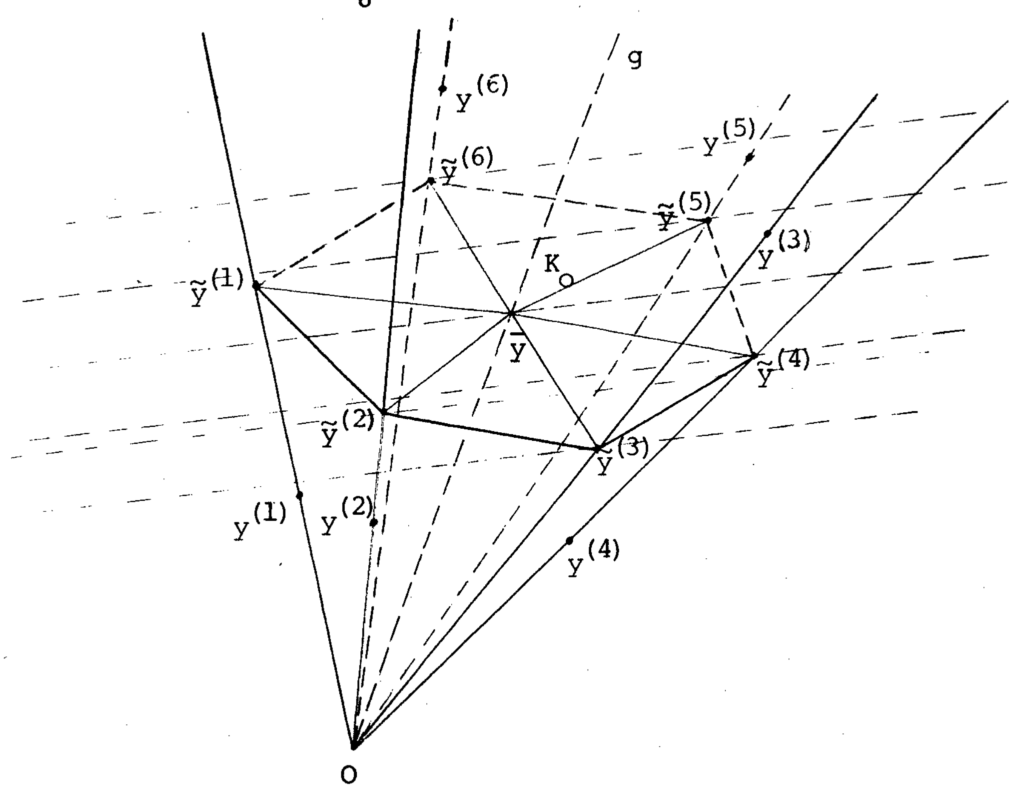

Fig. 7.1. Cone Y_o with cone-generators $y^{(\ell)}$, $\ell=1,\ldots,\ell=1,\ldots,6$, cone-axis g, hyperplane E_o and convex polyhedron $K_o=\mathrm{conv}\{\tilde{y}^{(1)},\ldots,\tilde{y}^{(6)}\}$.

Let E_o denote the hyperplane

$$(y-\bar{y})'\bar{y} = 0 \iff y'\bar{y} = ||\bar{y}||^2 \tag{37.1}$$

through $\bar{y}$ and orthogonal to axis g. Consider then the points $\tilde{y}^{(\ell)} \in Y_o$, $\ell=1,\dots,2n$, lying on E_o and on the straight lines through 0 and $y^{(\ell)}$, $\ell=1,\dots,2n$, hence,

$$\tilde{y}^{(\ell)} := \frac{||\bar{y}||^2}{y^{(\ell)\prime}\bar{y}} y^{(\ell)}, \ \ell=1,\dots,2n, \tag{38}$$

see Figure 7.1. Obviously, we get

$$y^{(\ell)\prime}\bar{y} > 0, \ \ell=1,\dots,2n, \tag{38.1}$$

as a condition for $\bar{y}$. Since the equation $\sum_{k=1}^{n} \alpha_k y^{(k)} = 0$, $\alpha_k \geq 0$, $k=1,\dots,2n$, cf. (32),(32.1), has no nonzero solution α, according to Gordan's transposition theorem, system (38.1) has solutions $\bar{y} \neq 0$. Let

$$K_o := \text{conv}\{\tilde{y}^{(1)}, \tilde{y}^{(2)}, \dots, \tilde{y}^{(2n)}\}$$

denote the convex polyhedron on E_o generated by the points $\tilde{y}^{(\ell)}$, $\ell=1,\dots,2n$. By (38.1) and (38.2) we get

$$y'\bar{y} > 0 \text{ for all } y \in K_o. \tag{38.1a}$$

According to (38) and (38.1), the cone Y_o can be represented by

$$Y_o = \bigcup_{\lambda \geq o} \lambda K_o = \{\lambda y: \lambda \geq 0, y \in K_o\}. \tag{38.3}$$

Based on the above representation of Y_o, approximations $\tilde{Y}_o$ of Y_o of the type (36) can be obtained by replacing the generating polyhedron K_o in Y_o by suitable approximations, e.g., ellipsoides, spheres, denoted by $\tilde{K}_o$:

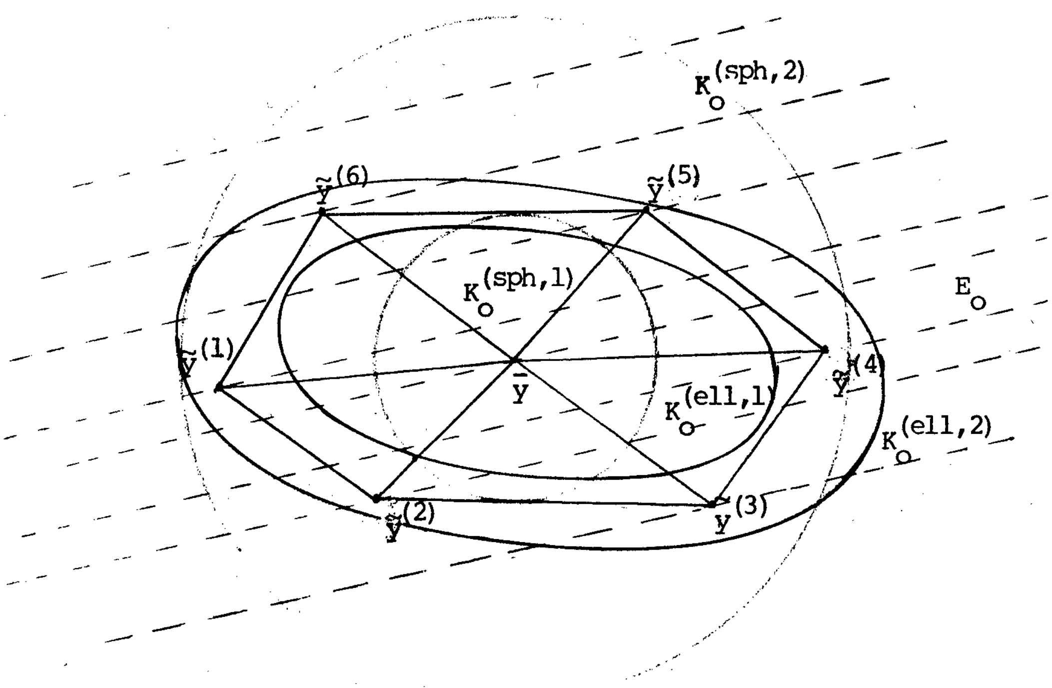

Fig. 7.2. Convex polyhedron K_o in hyperplane E_o and (second order) approximnations $\tilde{K}_o = K_o^{(ell,j)}$, $\tilde{K}_o = K_o^{(sph,j)}$, j=1,2

a) Ellipsoidal approximations: According to (38.1a), an outer ellipsoidal approximation $K_o^{(ell,2)} \supset K_o$ can be defined, for j=2, by

$$K_o^{(e\ell\ell,j)} := \{y \in E_o : y'\bar{y} > 0,\ (y-\bar{y})'\Gamma^{(j)}\Gamma^{(j)'}(y-\bar{y}) \le 1\}, \tag{39}$$

where the (n+m)×(n+m) matrix $\Gamma^{(2)} = \Gamma$ is chosen such that

$$\begin{aligned} &(\tilde{y}^{(\ell)}-\bar{y})'\Gamma\Gamma'(\tilde{y}^{(\ell)}-\bar{y}) \le 1,\ \ell=1,\dots,2n \\ &y'\bar{y} = 0 \Longrightarrow (\Gamma y)'\bar{y} = 0 \\ &\sum_{\ell=1}^{2n} (1-(\tilde{y}^{(\ell)}-\bar{y})'\Gamma\Gamma'(\tilde{y}^{(\ell)}-\bar{y})) \longrightarrow \min. \end{aligned} \tag{39.1}$$

Conversely, an "inner" ellipsoidal approximation $K_o^{(ell,1)}$ of K_o can be determined, cf. (39), by choosing a matrix $\Gamma^{(1)} = \Gamma$ such that

$$\begin{aligned} &(\tilde{y}^{(\ell)}-\bar{y})'\Gamma\Gamma'(\tilde{y}^{(\ell)}-\bar{y}) \ge 1,\ \ell=1,\dots,2n \\ &y'\bar{y} = 0 \Longrightarrow (\Gamma y)'\bar{y} = 0 \\ &\sum_{\ell=1}^{2n} (\tilde{y}^{(\ell)}-\bar{y})'\Gamma\Gamma'(\tilde{y}^{(\ell)}-\bar{y})-1 \longrightarrow \min \end{aligned} \tag{39.2}$$

Note that $K_o^{(ell,1)} \setminus K_o \neq \emptyset$ may happen.

b) Spherical approximations: An outer spherical approximation $K_o^{(sph,2)} \supset K_o$ is defined, cf. (38.1a), by

$$K_o^{(sph,j)} := \{y \in E_o :\ y'\bar{y} > 0,\ ||y-\bar{y}|| < \rho_j\}, \tag{40}$$

j=2, where the radius ρ_2 is given by

$$\rho_2 := \max_{1\le\ell\le 2n} ||\tilde{y}^{(\ell)}-\bar{y}|| \tag{40.1}$$

and $||.||$ denotes the Euclidean norm. Likewise, an "inner" spherical approximation $K_o^{(sph,1)}$ of K_o can be determined, cf. (40), by choosing the radius

$$\rho_1 := \min_{1\le\ell\le 2n} ||\tilde{y}^{(\ell)}-\bar{y}||, \tag{40.2}$$

where, as for $K_o^{(ell,1)}$, the relation $K_o^{(sph,1)} \setminus K_o \neq \emptyset$ is not excluded in general. Of course, an inner spherical approximation $K_o^{(sph,1)} \subset K_o$ is obtained by selecting the radius

$$\rho_1' := \min\{||y-\bar{y}|| :\ y \in \partial K_o\}, \tag{40.3}$$

where ∂K_o denotes the boundary of K_o.

Having an approximation $\tilde{K}_o$ of K_o as described by (39) and (40), the cone Y_o can be approximated now by the cone

$$\tilde{Y}_o := \{\lambda y :\ \lambda \ge 0,\ y \in \tilde{K}_o\}. \tag{41}$$

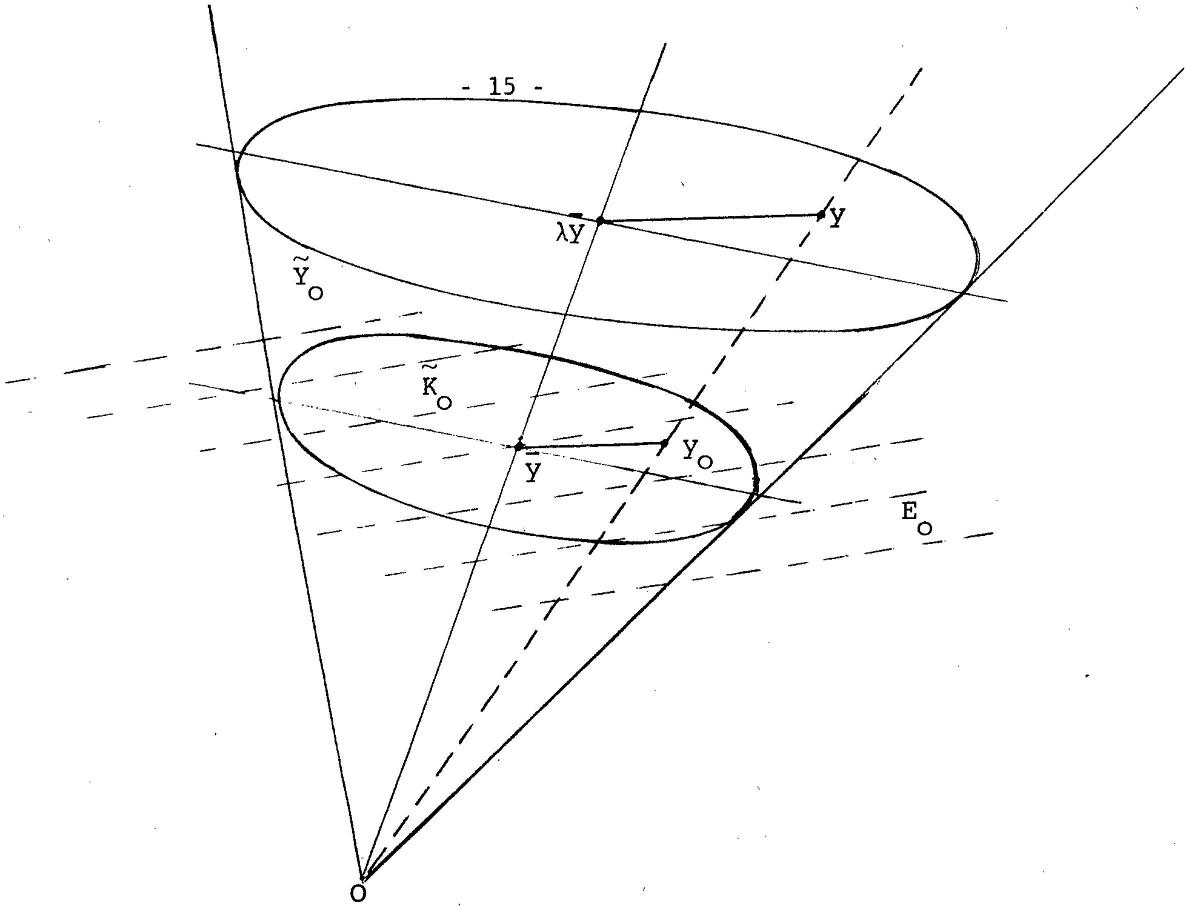

Fig. 7.3. Approximating cone $\tilde{Y}_o$ generated by $\tilde{K}_o$

For any point $y \in \mathbb{R}^{m+n}$, $y \neq 0$, the intersection y_o of the hyperplane E_o and the straight line through 0 and y is given by

$$y_o := \frac{||\bar{y}||^2}{y'\bar{y}}\, y. \tag{41.1}$$

Hence, according to the definition (41) of $\tilde{Y}_o$ we have that $y \in \tilde{Y}_o$, $y \neq 0$, if and only if the following simple relations for y hold:

$$y'\bar{y} > 0 \tag{42}$$

$$||\Gamma'(||\bar{y}||^2 y - y'y\ \bar{y})|| \le y'\bar{y} \tag{42.1a}$$

$$||y|| \le \frac{(\rho^2 + ||\bar{y}||^2)^{1/2}}{||\bar{y}||^2}\, y'\bar{y}, \tag{42.1b}$$

where $\Gamma = \Gamma^{(1)}, \Gamma^{(2)}$ and $\rho = \rho_1, \rho_2$. Obviously, (42)-(42.1b) are convex conditions for y.

Having an approximation $\tilde{Y}_o$ of Y_o, the probability function $P = P(\lambda, X)$ given by (33.1) can be approximated then, cf. (36.2), by

$$\tilde{P}(\lambda, X) := P\left(\begin{pmatrix} CF^U(\omega, X) - \lambda R_o(\omega) \\ F^U(\omega, X) - F^L(\omega, X) \end{pmatrix} \in \tilde{Y}_o\right). \tag{43}$$

Hence, in the above case from (42)-(42.1b) we obtain

$$\tilde{P}(\lambda, X) = P(||\Gamma'(||\bar{y}||^2 y(\omega) - \bar{y}'y(\omega)\ \bar{y})|| \le y(\omega)'\bar{y}), \tag{43.1a}$$

$$\tilde{P}(\lambda, X) = P(||y(\omega)|| \le \frac{(\rho^2 + ||\bar{y}||^2)^{1/2}}{||\bar{y}||^2}\, y(\omega)'\bar{y}), \tag{43.1b}$$

resp., where

$$y(\omega):=\begin{pmatrix} CF^{U}(\omega,X) - \lambda R_{o}(\omega) \\ F^{U}(\omega,X) - F^{L}(\omega,X) \end{pmatrix}. \tag{43.2}$$

Finally, we have to determine the axis $g=\{\lambda\bar{y}: \lambda\geq 0\}$ by selecting a generating vector $y\in Y_o$, $\bar{y}\neq 0$. Having $\bar{y} = \sum_{k=1}^{2n} \alpha_k y^{(k)}$ with $\alpha_k\geq 0$, $k=1,\dots,2n$, condition (38.1) can be fulfilled by choosing the coefficients $\alpha=(\alpha_1,\dots,\alpha_{2n})'$ such that

$$Y_o^M\alpha > 0, \tag{44}$$

where the $2n\times 2n$ matrix Y_o^M is given by

$$Y_o^M := (y^{(\ell)\prime}y^{(k)})_{\ell,k=1,\dots,2n} \tag{44.1}$$

$$= \left(\begin{array}{cccc|c} ||c_1||^2+1 & c_1'c_2 & \dots & c_1'c_n & \\ c_2'c_1 & ||c_2||^2+1 & \dots & c_2'c_n & \\ \vdots & \vdots & & \vdots & I_n \\ c_n'c_1 & c_n'c_2 & \dots & ||c_n||^2+1 & \\ \hline & & & & \\ & & I_n & & I_n \\ & & & & \end{array}\right),$$

where I_n is the $n\times n$ identity matrix. Thus, we may select then α such that $\min_{1\leq\ell\leq 2n} y^{(\ell)\prime}\bar{y}$ is maximized, which yields the linear program

$$\max\ t \tag{45}$$

$$\left(\begin{array}{c|c} -Y_o^M & 1 \\ \hline I_{2n} & 0 \end{array}\right)\binom{\alpha}{t} \leq \binom{0}{a_o} \tag{45.1}$$

$$\alpha\geq 0,\ t\geq 0, \tag{45.2}$$

where $a_o = (a_{o1}, a_{o2},\dots,a_{o2n})'$ is a given 2n-vector having positive components, and the constraint $\alpha\leq a_o$ is imposed because the direction of $\bar{y}$ is needed only.

On the other hand, for any vector $\bar{y}$, $\bar{y}\neq 0$, we consider the hyperplane E_o defined by (37.1) and the points $\tilde{y}^{(\ell)}$, $\ell=1,\dots,2n$, on E_o defined by (38):

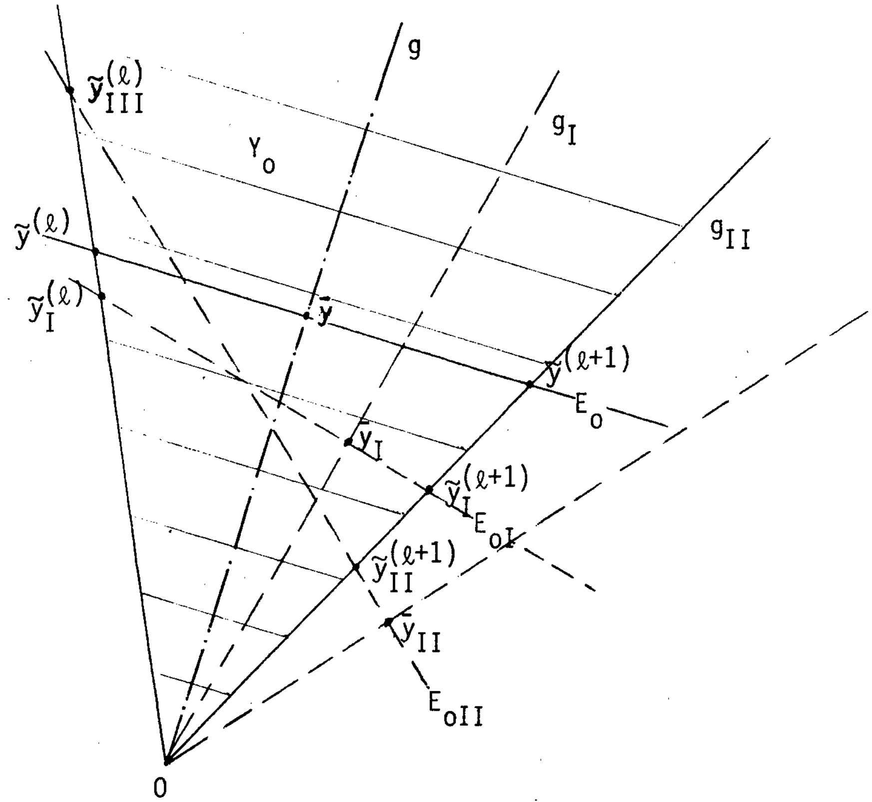

Fig. 7.4. Construction of a cone axis g of Y_o: Approximative generators $\bar{y}_I$, $\bar{y}_{II}$.

Since the axis $g=\{\lambda\bar{y}: \lambda\geq 0\}$ should pass through the center of the cone Y_o, the deviations between $\bar{y}$ and the points $\tilde{y}^{(\ell)}$, $\ell=1,\dots,2n$, should be well-balanced. Thus, $\bar{y}$ is choosen such that the quantity

$$\max_{1\leq\ell\leq 2n} ||\bar{y}-\tilde{y}^{(\ell)}|| = \max_{1\leq\ell\leq 2n} \left(\frac{||\bar{y}||^4}{(y^{(\ell)\prime}\bar{y})^2}||y^{(\ell)}||^2 - ||\bar{y}||^2 \right)^{1/2}, \tag{46}$$

cf. (38), is minimized, which yields, see (38.1), the optimization problem

$$\min \max_{1\leq\ell\leq 2n} ||\bar{y}-\tilde{y}^{(\ell)}|| \tag{47}$$

s.t.

$$y^{(\ell)\prime}\bar{y} > 0,\ \ell=1,\dots,2n \tag{47.1}$$

$$\bar{y}\in Y_o,\ ||\bar{y}||=1; \tag{47.2}$$

since only the direction of $\bar{y}$ is relevant for our purposes, the norm constraint $||\bar{y}||=1$ is added. According to (46), problem (47) is equivalent to the program

$$\max t \tag{47)'}$$

s.t.

$$t - \frac{y^{(\ell)\prime}\bar{y}}{||y^{(\ell)}||} \leq 0,\ \ell=1,\dots,2n \tag{47.1)'}$$

$$t\geq 0,\ \bar{y}\in Y_o,\ ||\bar{y}||=1. \tag{47.2)'}$$

Note that

$$\frac{y^{(\ell)'}\bar{y}}{||y^{(\ell)}||} = \cos \sphericalangle(y^{(\ell)},\bar{y}),$$

is the cosine of the angle between the vectors $y^{(\ell)}$ and $\bar{y}$.

Representing $\bar{y} \in Y_o$ by $\bar{y} = \sum_{k=1}^{2n} \alpha_k y^{(k)}$, $\alpha_k \geq 0$, $k=1,\dots,2n$, we observe that (47)' is closely related to (45).

8. Sensitivity analysis of probabilities of survival/failure

In the following we consider the sensitivity of the probability of survival/failure with respect to the variables (λ,X,z), i.e., with respect to the m_R-vector λ of deterministic load factors, the r-vector X of design variables, and the (n-m)-vector z of deterministic redundants in the n-vector $F=F(\omega)$ of member (element) loads. As developed in [8-13], derivatives of the probability functions $P=P(\lambda,X)$, $\tilde{P}=\tilde{P}(\lambda,X,z)$ can be obtained - under weak mathematical assumptions - by means of the **Transformation Method** in combination with a **stochastic completion technique**.

8.1. The probability functions (10.3)-(10.5). In order to show the differentiation of the probability functions given by (10.3)-(10.5) it is sufficient to consider probability functions of the type

$$P(\lambda,X) := P(\lambda R_o(\omega) \in [A(X)_d\sigma^L(\omega),\ A(X)\sigma^U(\omega)]), \tag{48}$$

where we suppose that the random vectors $R_o(\omega)$, $(\sigma^L(\omega),\sigma^U(\omega))$ have sufficiently smooth probability densities $\varphi=\varphi(R_o)$, $\psi=\psi(\sigma^L,\sigma^U)$ on $\mathbb{R}^m,\mathbb{R}^{2n}$, respectively. Hence, using also (33), we have that

$$P(\lambda,X) = P(\lambda,A),\ A=A(X) = (A_1(X),\dots,A_n(X))', \tag{48.1}$$

cf. (2.3), where

$$\begin{aligned} P(\lambda,A) :&= \int\limits_{\lambda R_o \in C[A_d\sigma^L,A_d\sigma^U]} \varphi(R_o)\psi(\sigma^L,\sigma^U)\,dR_o\,d\sigma^L d\sigma^U \\ &= \int\limits_{\begin{pmatrix} CA_d\sigma^U-\lambda R_o \\ A_d\sigma^U-A_d\sigma^L \end{pmatrix} \in Y_o} \varphi(R_o)\psi(\sigma^L,\sigma^U)\,dR_o\,d\sigma^L d\sigma^U. \end{aligned} \tag{48.2}$$

Applying, for given variables (λ,A) with $\lambda \neq 0$, $A_i>0$, $i=1,\dots,n=n_o$, resp., to (48.2) the transformation $T_{(\lambda,A)}: (R,F^L,F^U) \longrightarrow (R_o,\sigma^L,\sigma^U)$ in $\mathbb{R}^m \times \mathbb{R}^{2n}$ defined by

$$R_o := \frac{1}{\lambda}R,\ \sigma^L := A_d^{-1}F^L,\ \sigma^U := A_d^{-1}F^U, \tag{48.3}$$

we obtain

$$P(\lambda,A) = \int_{R\in C[F^L,F^U]} \varphi(\frac{1}{\lambda}R)\psi(A_d^{-1}F^L,A_d^{-1}F^U) \times \frac{1}{|\lambda|^m}\prod_{j=1}^{n}\frac{1}{A_j^2}\,dRdF^LdF^U. \tag{48.4}$$

Under weak assumptions [10], [13] we may interchange in (48.4) differentiation and integration, hence,

$$\frac{\partial P}{\partial\lambda}(\lambda,A) = -\frac{1}{\lambda}\int_{R\in C[F^L,F^U]} (\nabla\varphi(\frac{R}{\lambda})'\frac{R}{\lambda} + m\varphi(\frac{R}{\lambda}))\psi(A_d^{-1}F^L,A_d^{-1}F^U) \times \frac{1}{|\lambda|^m}\prod_{j=1}^{n}\frac{1}{A_j^2}\,dRdF^LdF^U. \tag{49}$$

Using the inverse transformation $T^{-1}_{(\lambda,A)}$, (49) yields

$$\begin{aligned}\frac{\partial P}{\partial\lambda}(\lambda,A) &= -\frac{1}{\lambda}\cdot\int_{\lambda R_o\in C[A_d\sigma^L,A_d\sigma^U]} \operatorname{div}(R_o\varphi(R_o))\psi(\sigma^L,\sigma^U)dR_od\sigma^Ld\sigma^U \\ &= -\frac{1}{\lambda}\int \operatorname{div}(R_o\varphi(R_o))P(\lambda,A|R_o)dR_o,\end{aligned} \tag{49.1}$$

with the conditional probability function

$$P(\lambda,A|R_o) := P(\lambda R_o \in C[A_d\sigma^L(\omega),A_d\sigma^U(\omega)]),\ R_o\in\mathbb{R}^m. \tag{49.2}$$

Considering now the derivative $\frac{\partial P}{\partial X_k}(\lambda,X)$ with respect to a design variable X_k, $1\le k\le r$, by means of (48.1) we find

$$\frac{\partial P}{\partial X_k}(\lambda,X) = \sum_{j=1}^{n}\frac{\partial P}{\partial A_j}(\lambda,A)\frac{\partial A_j}{\partial x_k}(X), \tag{50}$$

and, using again (48.4), by interchanging differentiation and integration there we get

$$\begin{aligned}\frac{\partial P}{\partial A_j}(\lambda,A) = -\frac{1}{A_j}\int_{R\in C[F^L,F^U]} &\varphi(\frac{R}{\lambda})[2\psi(A_d^{-1}F^L,A_d^{-1}F^U) \\ &+ \frac{\partial\psi}{\partial\sigma_j^L}(A_d^{-1}F^L,A_d^{-1}F^U)\frac{F_j^L}{A_j} + \frac{\partial\psi}{\partial\sigma_j^U}(A_d^{-1}F^L,A_d^{-1}F^U)\frac{F_j^U}{A_j}] \\ &\times\frac{1}{|\lambda|^m}\prod_{j=1}^{n}\frac{1}{A_j^2}\,dRdF^LdF^U.\end{aligned} \tag{51}$$

Using the inverse $T^{-1}_{(\lambda,A)}$ of (48.3) again, from (50.1) we get

$$\begin{aligned}\frac{\partial P}{\partial A_j}(\lambda,A) &= -\frac{1}{A_j}\int_{\lambda R_o\in C[A_d\sigma^L,A_d\sigma^U]} \varphi(R_o)[2\psi(\sigma^L,\sigma^U) \\ &\quad + \frac{\partial\psi}{\partial\sigma_j^L}(\sigma^L,\sigma^U)\sigma_j^L + \frac{\partial\psi}{\partial\sigma_j^U}(\sigma^L,\sigma^U)\sigma_j^U]dR_od\sigma^Ld\sigma^U \\ &= -\frac{1}{A_j}\int \operatorname{div}((\sigma^L,\sigma^U)^{(j)}\psi(\sigma^L,\sigma^U))P(\lambda,A|\sigma^L,\sigma^U)d\sigma^Ld\sigma^U,\end{aligned} \tag{51.1}$$

with the conditional probability function

$$P(\lambda,A|\sigma^L,\sigma^U) := P(\lambda R_o(\omega)\in C[A_d\sigma^L,A_d\sigma^U]),\sigma^L,\sigma^U\in\mathbb{R}^n, \tag{51.2}$$

and

$$(\sigma^L,\sigma^U)^{(j)} := (0,\ldots,0,\sigma_j^L,0,\ldots,0,\sigma_j^U,0,\ldots,0)', \tag{51.3}$$

where σ_j^L,σ_j^U are placed at the j-th, (n+j)-th position, respectively.

According to the representation (31) of the event $\{R\in C[F^L,F^U]\}$, the conditional probability functions $P(\lambda,A|R_o)$, $P(\lambda,A|\sigma^L,\sigma^U)$ can be represented, cf. (31)-(33.1),(48.2), by

$$\begin{aligned} P(\lambda,A|R_o) &= P(y(\omega|R_o) \in Y_o), \\ P(\lambda,A|\sigma^L,\sigma^U) &= P(y(\omega|\sigma^L,\sigma^U) \in Y_o), \end{aligned} \tag{52}$$

where

$$\begin{aligned} y(\omega|R_o) &:= \begin{pmatrix} CA_d\sigma^U(\omega) - \lambda R_o \\ A_d\sigma^U(\omega)) - A_d\sigma^L(\omega) \end{pmatrix}, \\ y(\omega|\sigma^L,\sigma^U) &:= \begin{pmatrix} CA_d\sigma^U - \lambda R_o(\omega) \\ A_d\sigma^U - A_d\sigma^L \end{pmatrix}. \end{aligned} \tag{52.1}$$

Hence, corresponding to the approximation $\tilde{P}(\lambda,A)$ of $P(\lambda,A)$, cf. (43)-(43.2), the above conditional probability functions (52) can be approximated by

$$\begin{aligned} \tilde{P}(\lambda,A|R_o) &:= P(y(\omega|R_o) \in \tilde{Y}_o), \\ \tilde{P}(\lambda,A|\sigma^L,\sigma^U) &:= P(y(\omega|\sigma^L,\sigma^U) \in \tilde{Y}_o), \end{aligned} \tag{52.2}$$

where $\tilde{Y}_o$ is the approximation of the convex cone as described in Section 7.

By a slight modification in the equations (49.1),(51.1), resp., the derivatives of $P=P(\lambda,A)$ can also be represented by means of **expectations**:

Theorem 8.1. Under the assumptions in Section 8 we get

$$\begin{aligned} \text{a)}\ \frac{\partial P}{\partial \lambda}(\lambda,A) &= -\frac{1}{\lambda} E \frac{\operatorname{div}(R_o(\omega)\varphi(R_o(\omega)))}{\varphi(R_o(\omega))} 1_{C[A_d\sigma^L(\omega),A_d\sigma^U(\omega)]}(\lambda R_o(\omega)) \\ &= -\frac{1}{\lambda} E \frac{\operatorname{div}(R_o(\omega)\varphi(R_o(\omega)))}{\varphi(R_o(\omega))} P(\lambda,A|R_o(\omega)), \end{aligned} \tag{53}$$

$$\begin{aligned} \text{b)}\ \frac{\partial P}{\partial A_j}(\lambda,A) &= -\frac{1}{A_j} E \frac{\operatorname{div}((\sigma^L(\omega),\sigma^U(\omega))^{(j)}\psi(\sigma^L(\omega),\sigma^U(\omega)))}{\psi(\sigma^L(\omega),\sigma^U(\omega))} \\ &\qquad \times 1_{C[A_d\sigma^L(\omega),A_d\sigma^U(\omega)]}(\lambda R_o(\omega)) \\ &= -\frac{1}{A_j} E \frac{\operatorname{div}((\sigma^L(\omega),\sigma^U(\omega))^{(j)}\psi(\sigma^L(\omega),\sigma^U(\omega)))}{\psi(\sigma^L(\omega),\sigma^U(\omega))} P(\lambda,A|\sigma^L(\omega),\sigma^U(\omega)). \end{aligned} \tag{53.1}$$

Remark 8.1.

a) Selecting fixed variables $(\bar{\lambda},\bar{A})$ such that $\bar{\lambda}\neq 0$, $\bar{A}_i>0$, $i=1,\ldots,n$, and applying then - instead of $T^{-1}_{(\lambda,A)}$ - the inverse transformation $T^{-1}_{(\bar{\lambda},\bar{A})}$ to (49), we obtain

$$\frac{\partial P}{\partial \lambda}(\lambda,A) = -\frac{1}{\lambda} \int_{\bar{\lambda}R_o \in C[\bar{A}_d\sigma^L,\bar{A}_d\sigma^U]} (\nabla\varphi(\frac{\bar{\lambda}}{\lambda}R_o)'\frac{\bar{\lambda}}{\lambda}R_o + m\varphi(\frac{\bar{\lambda}}{\lambda}R_o))$$

$$\times \psi(A_d^{-1}\bar{A}_d\sigma^L, A_d^{-1}\bar{A}_d\sigma^U)\,|\frac{\bar{\lambda}}{\lambda}|^m \prod_{j=1}^{n} (\frac{\bar{A}_j}{A_j})^2 dR_o d\sigma^L d\sigma^U,$$

and $\frac{\partial P}{\partial A}(\lambda,A)$ can be represented in the same way. Hence, the derivatives $\frac{\partial P}{\partial \lambda}$, $\frac{\partial P}{\partial A_j}$ may be represented by integrals over the **fixed** domain $B_{(\bar{\lambda},\bar{A})} :=$ $\{(R_o,\sigma^L,\sigma^U): \bar{\lambda}R_o \in C[\bar{A}_d\sigma^L,\bar{A}_d\sigma^U]\}$ in the (R_o,σ^L,σ^U)-space.

b) Since the domain of integration in the integral representation (49), (51), resp., of $\frac{\partial P}{\partial \lambda}$, $\frac{\partial P}{\partial A_j}$ is independent of the variables λ, $A=(A_1,\dots,A_n)'$, the higher order derivatives - of arbitrary order - of $P=P(\lambda,A)$ can be obtained by further differentiation of the equations (49),(51) with respect to $\lambda, A_j, 1\le j\le n$.

c) Having the mean value representations (53),(53.1), gradient estimates - as well as estimates of the probability function itself - can be obtained by a suitable sampling procedure.

8.2 The probability function (18.1). Corresponding to (48)-(48.2) we note first that

$$\tilde{P}(X,z) = \tilde{P}(A,z),\ A=A(X) = (A_1(X),\dots,A_n(X))', \tag{54}$$

cf. (2.3), where

$$\tilde{P}(A,z) := P\begin{pmatrix} A_{I,d}\sigma_I^L(\omega) \le C_I^{-1}(R(\omega)-C_{II}z) \le A_{I,d}\sigma_I^U(\omega) \\ A_{II,d}\sigma_{II}^L(\omega) \le z \le A_{II,d}\sigma_{II}^U(\omega) \end{pmatrix}. \tag{54.1}$$

Supposing that the random vectors $R(\omega),(\sigma^L(\omega),\sigma^U(\omega))$ have sufficiently smooth densities $\varphi=\varphi(R)$, $\psi=\psi(\sigma^L,\sigma^U)$ on $\mathbb{R}^m,\mathbb{R}^{2n}$, resp., and defining here the integral transformation $T_{(z,A)}: (S,F_I^L,q_{II}^L,F_I^U,q_{II}^U) \longrightarrow (R,\sigma^L,\sigma^U)$ by

$$\begin{aligned} R &:= C_{II}z + S \\ \sigma_I^L &:= A_{I,d}{}^{-1}F_I^L,\ \sigma_{II}^L := A_{II,d}{}^{-1}(z+q_{II}^L) \\ \sigma_I^U &:= A_{I,d}{}^{-1}F_I^U,\ \sigma_{II}^U := A_{II,d}{}^{-1}(z+q_{II}^U), \end{aligned} \tag{54.2}$$

where $A_j>0$, $1\le j\le n$, we find

$$\tilde{P}(A,z) = \int_{\substack{F_I^L\le C_I^{-1}S\le F_I^U \\ q_{II}^L\le 0\le q_{II}^U}} \varphi(C_{II}z+S)\psi\left(\begin{pmatrix} A_{I,d}{}^{-1}F_I^L \\ A_{II,d}{}^{-1}(z+q_{II}^L)\end{pmatrix}, \begin{pmatrix} A_{I,d}{}^{-1}F_I^U \\ A_{II,d}{}^{-1}(z+q_{II}^U)\end{pmatrix}\right) \times \prod_{j=1}^{n} \frac{1}{A_j^2} dS dF_I^L dq_{II}^L dF_I^U dq_{II}^U. \tag{54.3}$$

Having (54.3), the derivatives of $\tilde{P}=\tilde{P}(A,z)$ - of various orders - follow again under weak assumptions [10],[13] by interchanging differentiation

and integration in (54.3). Hence, corresponding to (51)-(51.3), for $j=1,2,\dots,n$ we find

$$\frac{\partial\tilde{P}}{\partial A_j}(A,z) = -\frac{1}{A_j}\int_{A_{II,d}\sigma^L_{II}\le z\le A_{II,d}\sigma^U_{II}} \operatorname{div}((\sigma^L,\sigma^U)^{(j)}\psi(\sigma^L,\sigma^U)) \times P^I(A,z|\sigma^L,\sigma^U)\,d\sigma^L d\sigma^U, \tag{55}$$

where $P^I = P^I(A,z|\sigma^L,\sigma^U)$ is the conditional probability function given by

$$P^I(A,z|\sigma^L,\sigma^U) := P(A_{I,d}\sigma^L_I \le C_I^{-1}(R(\omega)-C_{II}z) \le A_{I,d}\sigma^U_I). \tag{55.1}$$

Moreover, for $j\in\{j_1:\ 1\le l\le n-m\}$ we get

$$\frac{\partial\tilde{P}}{\partial z_j}(A,z) = \int\nabla\varphi(R)'c_j\tilde{P}(A,z|R)\,dR + \frac{1}{A_j}\int_{A_{II,d}\sigma^L_{II}\le z\le A_{II,d}\sigma^U_{II}} \left(\frac{\partial\psi}{\partial\sigma^L_j}(\sigma^L,\sigma^U) + \frac{\partial\psi}{\partial\sigma^U_j}(\sigma^L,\sigma^U)\right)P^I(A,z|\sigma^L,\sigma^U)\,d\sigma^L d\sigma^U, \tag{55.2}$$

where

$$\tilde{P}(A,z|R) := P\begin{pmatrix} A_{I,d}\sigma^L_I(\omega) \le C_I^{-1}(R-C_{II}z) \le A_{I,d}\sigma^U_I(\omega) \\ A_{II,d}\sigma^L_{II}(\omega) \le z \le A_{II,d}\sigma^U_{II}(\omega)\end{pmatrix}. \tag{55.3}$$

Remark 8.2.

Using the inverse transformation $T_{(\bar{z},\bar{A})}^{-1}$ with given fixed variables $\bar{z}$, $\bar{A}$, cf. Remark 8.1, we also obtain integral representations of the derivatives having a **fixed** domain of integration in the space of the original (R,σ^L,σ^U)-variables.

8.3. The probability function (26.1)

According to the definition of $F^L(\omega,X), F^U(\omega,X)$ given in Section 1 for different cases, the probability function (26.1) can be represented by

$$P(\lambda,X) = P(a(\omega)'V(\lambda,X)\delta \le 0 \text{ for all } \delta\in\Delta_o), \tag{56}$$

where

$$a(\omega) := \begin{pmatrix} R_o(\omega) \\ \sigma^U(\omega) \\ \sigma^L(\omega)\end{pmatrix}, \quad \delta := \begin{pmatrix} u \\ \bar{u}^+ \\ \bar{u}^-\end{pmatrix}, \tag{56.1}$$

Δ_o is the convex polyhedron of elements δ represented by the system of linear equalities/inequalities (23.1)-(23.3), and

$$V = V(\lambda,X) = V(\lambda,A(X),\ W^{\bar{y}}(X),\ W^{\bar{z}}(X),\ W^{\bar{p}}(X)) \tag{56.1}$$

is an $(m+2n)\times(m+2n)$ matrix given analytically; in case of trusses we have that

$$V(\lambda,X) := \left(\begin{array}{c:c:c} \lambda I & & \\ \hdashline & -A_d(x) & \\ \hdashline & & A_d(X) \end{array}\right). \tag{56.2}$$

Having the extreme points $\delta^{(\ell)}$, $\ell=1,\dots,\ell_o$, of Δ_o, we also get

$$P(\lambda,X) = P(a(\omega)'V(\lambda,X)\delta^{(\ell)} \le 0,\ \ell=1,\dots,\ell_o). \tag{56'}$$

Since the number ℓ_o of extreme points of Δ_o may be very large, and the numerical computation of $\delta^{(\ell)}$, $\ell=1,\ldots,\ell_o$, is very time consuming in general, first we are looking for upper and lower bounds of $P=P(\lambda,X)$:

Considering an arbitrary sequence of elements

$\delta_1,\delta_2,\ldots,\delta_j,\ldots$ in Δ_o,

we find for any $\nu\in N$ the upper bounds

$$P_\nu(\lambda,X;\delta_1,\ldots,\delta_\nu):=P(a(\omega)'V(\lambda,X)\delta_j\leq 0,\ j=1,\ldots,\nu), \tag{57}$$

where

$$P_\nu(\lambda,X;\delta_1,\ldots,\delta_\nu)\geq P_{\nu+1}(\lambda,X;\delta_1,\ldots,\delta_{\nu+1})\geq P(\lambda,X). \tag{58}$$

for each $\nu=1,2,\ldots$. Obviously, **minimal upper bounds** $P^*_\nu(\lambda,X)$ are obtained by minimizing (57) with respect to $\delta_j\in\Delta_o$, $j=1,\ldots,\nu$, hence, we put

$$P^*_\nu(\lambda,X):=\min\{P_\nu(\lambda,X;\delta_1,\ldots,\delta_\nu):\ \delta_j\in\Delta_o,\ j=1,\ldots,\nu\}. \tag{59}$$

By means of optimum upper bounds the true probability $P(\lambda,X)$ can be reached in a finite number of steps:

Lemma 8.1. There is an integer $\nu_o=\nu_o(\lambda,X)$, $\nu_o\leq\ell_o$, such that

$$P^*_{\nu_o}(\lambda,X)=P(\lambda,X). \tag{60}$$

The assertion follows from the inequalities

$$\begin{aligned}P(\lambda,X) &= P_{\ell_o}(\lambda,X;\delta^{(1)},\ldots,\delta^{(\ell_o)})\\ &\leq P^*_\nu(\lambda,X)\leq P_\nu(\lambda,X;\delta^{(1)},\ldots,\delta^{(\nu)})\end{aligned}$$

for each $\nu=1,2,\ldots,\ell_o$.

According to [19], **suboptimal upper bounds** for $P(\lambda,X)$ can be obtained **iteratively** as follows:

Stage 1. Define

$$\tilde{P}^*_1(\lambda,X):=P^*_1(\lambda,X)=\min\{P_1(\lambda,X;\delta_1):\ \delta_1\in\Delta_o\}, \tag{61a}$$

and let $\delta^*_1=\delta^*_1(\lambda,X)$ denote an element of Δ_o such that

$$P^*_1(\lambda,X)=P_1(\lambda,X;\delta^*_1(\lambda,X)).$$

Stage ν. Having $\delta^*_j=\delta^*_j(\lambda,X)$, $j=1,\ldots,\nu-1$, for $\nu>1$ define

$$\tilde{P}^*_\nu(\lambda,X):=\min\{P_\nu(\lambda,X;\delta^*_1(\lambda,X),\ldots,\delta^*_{\nu-1}(\lambda,X),\delta_\nu):\ \delta_\nu\in\Delta_o\}, \tag{61b}$$

and denote by $\delta^*_\nu=\delta^*_\nu(\lambda,X;\delta^*_1,\ldots,\delta^*_{\nu-1})$ an optimal solution in (61b). Obviously we have that

$$\tilde{P}^*_\nu(\lambda,X)\geq P^*_\nu(\lambda,X)\geq P(\lambda,X),\ \nu=1,2,\ldots\ .$$

Clearly, a big advantage in (61b) is that we have only one single decision vector δ_ν, whereas in (59) we have to deal with ν decision vectors $\delta_1,\ldots,\delta_\nu$. On the other hand, with the suboptimal upper bounds $P^*_\nu(\lambda,X)$ the exact value $P(\lambda,X)$ can not be reached in general in a finite number of steps.

According to (56') we find, cf. (29),(29.1),

$$P(\lambda,X) = 1 - P(\mathbf{F}_1 \cup \ldots \cup \mathbf{F}_{\ell_o})$$

with the failure domains $\mathbf{F}_\ell$ given by

$$\mathbf{F}_\ell := \{\omega\in\Omega:\ a(\omega)'V(\lambda,X)\delta^{(\ell)}>0\},\quad \ell=1,\ldots,\ell_o.$$

Hence, **lower bounds** for $P(\lambda,X)$ follow by applying Bonferroni-bounds [10] to $P(\bigcup_{\ell=1}^{\ell_o} \mathbf{F}_\ell)$. These bounds can be described by means of an estimate $\hat{\ell}_o$ of ℓ_o and by probability functions of the type

$$Q_\nu(\lambda,X;\delta_1,\ldots,\delta_\nu) := P(a(\omega)'V(\lambda,X)\delta_j>0,\ j=1,\ldots,\nu) \tag{62}$$

similar to $P_\nu(\lambda,X;\delta_1,\ldots,\delta_\nu)$, $\delta_j\in\Delta_o$, $j=1,\ldots,\nu$.

E.g., for $\nu=1$ we get

$$\begin{aligned}P(\lambda,X) &\geq 1 - \sum_{\ell=1}^{\ell_o} P(a(\omega)'V(\lambda,X)\delta^{(\ell)} > 0)\\ &= 1 - \sum_{\ell=1}^{\ell_o} Q_1(\lambda,X;\delta^{(\ell)}) \geq 1 - \ell_o Q_1^*(\lambda,X),\end{aligned}$$

where, for $\nu=1,2,\ldots,$

$$Q_\nu^*(\lambda,X) := \max\{Q_1(\lambda,X;\delta_1,\ldots,\delta_\nu):\ \delta_j\in\Delta_o,\ 1\leq j\leq\nu\}. \tag{62.1}$$

Consequently, for the approximative computation of the probability $P(\lambda,X)$ we have to solve optimization problem of the type

$$\underset{\delta_j\in\Delta_o,1\leq j\leq\nu}{\text{minimize}}\quad P_\nu(\lambda,X;\delta_1,\ldots,\delta_\nu). \tag{63}$$

and

$$\underset{\delta_j\in\Delta_o,1\leq j\leq\nu}{\text{maximize}}\quad Q_\nu(\lambda,X;\delta_1,\ldots,\delta_\nu). \tag{63.1}$$

Moreover, for the approximative solution of reliability-oriented optimization problems of the type

$$\max_{x\in D} P(\lambda_o,X) \tag{64}$$

with a given load factor $\lambda_o\in\mathbb{R}$, we have the **maxmin-problem**

$$\underset{X\in D}{\text{maximize}}\quad \underset{\delta_j\in\Delta_o,1\leq j\leq\nu}{\text{minimize}}\quad P_\nu(\lambda_o,X;\delta_1,\ldots,\delta_\nu). \tag{64.1}$$

Thus, in each case the derivatives of the probability functions P_ν, Q_ν, resp., are needed.

By the transformation

$$\theta_j := V(\lambda,X)\delta_j,\quad j=1,\ldots,\nu, \tag{65}$$

for $P_\nu=P_\nu(\lambda,X;\delta_1,\ldots,\delta_\nu)$ we find the representation

$$P_\nu(\lambda,X;\delta_1,\ldots,\delta_\nu) = W_\nu(V(\lambda,X)\delta_1,\ldots,V(\lambda,X)\delta_\nu), \tag{66}$$

where

$$W_\nu(\theta_1,\ldots,\theta_\nu) := P(\theta_j' a(\omega) \leq 0,\ j=1,\ldots,\nu), \tag{67}$$

and Q_ν can be represented in the same way. Since the derivatives of $V=V(\lambda,X)$ can be obtained analytically, the remaining problem is the differentiation of W_ν.

8.3.1. Exact differentiation formulas in case of $\nu \le \dim a(.)$

In many practical cases the stage number ν is small, hence, the assumption

$\nu \le d:= \dim a(.)= m+2n$

is not too restrictive. For the computation of the partial derivative $\frac{\partial W_\nu}{\partial \theta_{jk}}$ for a given pair (j,k), $1\le j\le\nu$, $1\le k\le d$, we consider a partition $\Theta = (\Theta_I,\Theta_{II})$ of the $\nu\times d$ matrix

$$\Theta:= \begin{pmatrix} \theta_1' \\ \theta_2' \\ \vdots \\ \theta_\nu' \end{pmatrix} \tag{68}$$

such that

$$\begin{array}{ll} \text{i)} & \theta_{jk} \text{ is an element of } \Theta_I \text{ and} \\ \text{ii)} & \operatorname{rank} \Theta_I = \nu. \end{array} \tag{69}$$

Supposing then that the random vector $a=a(\omega)$ has a probability density $f=f(a)$, and partitioning the vector $a\in\mathbb{R}^d$ in the same way $a=(a_I,a_{II})$ as the matrix Θ, by the integral transformation

$$a = \begin{pmatrix} a_I \\ a_{II} \end{pmatrix} := \begin{pmatrix} \Theta_I^{-1} p_I \\ p_{II} \end{pmatrix} \tag{70}$$

we find

$$W_\nu(\theta_1,\ldots,\theta_\nu) = \int_{p_I+\Theta_{II}p_{II}\le 0} f(\Theta_I^{-1}p_I,p_{II})\frac{dp_I}{|\det\Theta_I|} . \tag{71}$$

Obviously, by means of this transformation, the domain of integration in (71) is now independent of the element θ_{jk} and - of course - also independent of all other elements $\theta_{\iota\kappa}$ contained in Θ_I. Thus, if the density f of $a(\omega)$ is sufficiently smooth the derivative follows, cf. [10],[13], by interchanging differentiation and integration:

Theorem 8.2. Under appropriate assumptions [10],[13] on the density $f=f(a)$ of $a(\omega)$, the partial derivative $\frac{\partial}{\partial\theta_{jk}}W_\nu$, is given by

$$\frac{\partial W_\nu}{\partial\theta_{jk}}(\theta_1,\ldots,\theta_\nu)= - \int_{\Theta a\le 0} (\nabla_{a_I} f(a)'(\Theta_I^{-1'})_j' a_k+f(a)(\Theta_I^{-1'})_{jk})da, \tag{72}$$

which can be represented also by the **expectation**

$$\frac{\partial W_\nu}{\partial\theta_{jk}}(\theta_1,\ldots,\theta_\nu)= -E\left(\left(\frac{\nabla a_I f(a(\omega))}{f(a(\omega))}\right)'(\Theta_I^{-1'})_j' a_k(\omega)+(\Theta_I^{-1'})_{jk})1_{[\Theta a\le 0]}(a(\omega)\right), \tag{72.1}$$

where $(\Theta_I^{-1'})_j$ denotes the j-th row of $\Theta_I^{-1'}$.

Corollary 8.1. The derivatives $\frac{\partial W_\nu}{\partial \theta_{\iota\kappa}}$ of W_ν with respect to elements $\theta_{\iota\kappa}$ of Θ_I have the same form.

8.3.2. Approximative derivatives of W_ν

If condition (71) can not be fulfilled, e.g. in the case $\nu > d$, then approximative derivatives of W_ν - of an arbitrary high accuracy - can be obtained by the following **stochastic completion technique** [8],[13]:

Let $z_j = z_j(\omega)$, $j=1,\dots,\nu$, denote real random variables such that

i) $z_j(\omega)$, $j=1,\dots,\nu$, and $a(\omega)$ are stochastically independent

ii) $Ez_j(\omega)=0$, and $z_j(\omega)$ has a continuous probability density $\psi_j = \psi_j(t)$, $j=1,\dots,\nu$. (73)

By means of the "stochastic completion terms" $z_j = z_j(\omega)$, $j=1,\dots,\nu$, for W_ν we obtain the approximative probability function

$$\tilde{W}_\nu(\theta_1,\dots,\theta_\nu) := P(\theta_j' a(\omega) + z_j(\omega) \le 0,\ j=1,\dots,\nu). \tag{74}$$

We find

$$\begin{aligned}\tilde{W}_\nu(\theta_1,\dots,\theta_\nu) &= P(z_j(\omega) \le -\theta_j' a(\omega),\ j=1,\dots,\nu)\\ &= E \prod_{j=1}^{\nu} \Psi_j(-\theta_j' a(\omega)),\end{aligned} \tag{75}$$

where Ψ_j denotes the distribution function of $z_j(\omega)$; if $z_j(\omega)$ has the normal distribution $N(0,\sigma_j)$, then

$$\tilde{W}_\nu(\theta_1,\dots,\theta_\nu) = E \prod_{j=1}^{\nu} \Phi(-\frac{1}{\sigma_j}\theta_j' a(\omega)), \tag{75.1}$$

where Φ is the distribution function of $N(0,1)$.

The partial derivatives of $\tilde{W}_\nu$ read

$$\frac{\partial \tilde{W}_\nu}{\partial \theta_{jk}} = -E \prod_{\substack{\ell=1\\ \ell\neq j}}^{\nu} \Psi_\ell(-\theta_\ell' a(\omega)) \psi_j(-\theta_j' a(\omega)) a_k(\omega), \tag{76}$$

$j=1,\dots,\nu$, $k=1,\dots,d$, and higher order derivatives of $\tilde{W}_\nu$ can be obtained in the same way. If

$$z(\omega) = (z_1(\omega),\dots,z_\nu(\omega))' \longrightarrow 0 \text{ w.p.1} \tag{77}$$

(with probability one), then under some regularity assumptions [9],[13]

$$\begin{aligned}&\tilde{W}_\nu(\theta_1,\dots,\theta_\nu) \longrightarrow W_\nu(\theta_1,\dots,\theta_\nu),\\ &\frac{\partial \tilde{W}_\nu}{\partial \theta_{jk}}(\theta_1,\dots,\theta_\nu) \longrightarrow \frac{\partial W_\nu}{\partial \theta_{jk}}(\theta_1,\dots,\theta_\nu).\end{aligned} \tag{78}$$

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